TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 5, May 1995

INEQUALITIES FOR ZERO-BALANCED HYPERGEOMETRIC FUNCTIONS

G. D. ANDERSON, R. W. BARNARD, K. C. RICHARDS,
M. K. VAMANAMURTHY, AND M. VUORINEN

ABSTRACT. The authors study certain monotoneity and convexity properties of the Gaussian hypergeometric function and those of the Euler gamma function.

1. INTRODUCTION

The Gaussian *hypergeometric function* is defined by

$$F(a, b; c; x) = {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!}, \tag{1.1}$$

for $x \in (-1, 1)$, where (a, n) denotes the shifted factorial function $(a, n) = a(a+1)\cdots(a+n-1)$, $n = 1, 2, \ldots$, and $(a, 0) = 1$ for $a \neq 0$. The sum is well defined at least when $(c, n) \neq 0$, i.e., when $c \neq 0, -1, -2, \ldots$. This function has found frequent applications in various fields of the mathematical and natural sciences [Ask2]. Many elementary transcendental functions are special cases or limiting cases of $F(a, b; c; x)$; for an extensive list see [AS, pp. 556–566], [PBM, pp. 430–615]. Two important special cases are the complete elliptic integrals

$$\mathscr{K}(x) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right), \qquad \mathscr{E}(x) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; x^2\right). \tag{1.2}$$

As a function of its parameters a, b, c the function $F(a, b; c; x)$ is smooth, and it is thus natural to expect that the properties of $\mathscr{K}(x)$ extend also to $F(a, b; c; x^2)$ for (a, b, c) close to $(\frac{1}{2}, \frac{1}{2}, 1)$. Recall that $F(a, b; c; x)$ is called *zero-balanced* if $c = a + b$. We obtain several results for the zero-balanced $F(a, b; a+b; x)$, $a, b > 0$, extending well-known properties of $\mathscr{K}(x)$.

1.3. **Theorem.** (1) *For* $a, b \in (0, \infty)$ *the function*

$$f(x) \equiv \frac{1 - F(a, b; a+b; x)}{\log(1-x)}$$

Received by the editors September 17, 1993; originally communicated to the *Proceedings of the AMS* by Hal L. Smith.

1991 *Mathematics Subject Classification.* Primary 33C05, 33B15; Secondary 26D07.

Key words and phrases. Hypergeometric function, gamma function, elliptic integral.

0002-9947/95 \$1.00 + \$.25 per page

is strictly increasing from $(0, 1)$ *onto* $(ab/(a+b), 1/B)$, *where* $B = B(a, b)$ *is the Euler beta function.*

(2) *For* $a, b \in (0, \infty)$ *the function*

$$g(x) \equiv BF(a, b; a+b; x) + \log(1-x)$$

is strictly decreasing from $(0, 1)$ *onto* (R, B), *where*

$$R = -\Psi(a) - \Psi(b) - 2\gamma.$$

Here $\Psi(a) = \Gamma'(a)/\Gamma(a)$, *and* γ *is the Euler-Mascheroni constant.*

1.4. **Theorem.** *For* $a, b \in (0, \infty)$, *let*

$$f(x) = xF(a, b; a+b; x)/\log(1/(1-x))$$

on $(0, 1)$ *and let* B, R *be as in Theorem* 1.3.

(1) *If* $a, b \in (0, 1)$, *then* f *is decreasing with range* $(1/B, 1)$.

(2) *If* $a, b \in (1, \infty)$, *then* f *is increasing with range* $(1, 1/B)$.

(3) *If* $a = b = 1$, *then* $f(x) = 1$ *for all* $x \in (0, 1)$.

(4) *If* $a, b \in (0, 1)$ *the function* $g_1(x) \equiv BF(a, b; a+b; x) + (1/x)\log(1-x)$ *is increasing from* $(0, 1)$ *onto* $(B-1, R)$.

(5) *If* $a, b \in (1, \infty)$, *then* g_1 *is decreasing from* $(0, 1)$ *onto* $(R, B-1)$.

Here Theorem 1.3(1) generalizes the fact that $((2/\pi)\mathscr{K}(x) - 1)/\log(1/x')$ is increasing from $(0, 1)$ onto $(1/2, 2/\pi)$; 1.3(2) generalizes the well-known fact that $\mathscr{K}(x) + \log x'$ is decreasing from $(0, 1)$ onto $(\log 4, \pi/2)$ (cf. [AVV1, Theorem 2.2(2)]); and 1.4(1) generalizes the result that $x^2\mathscr{K}(x)/\log(1/x')$ is decreasing from $(0, 1)$ onto $(1, \pi)$ [AVV3, Theorem 2(19)].

The asymptotic relation

$$F(a, b; a+b; x) \sim -\frac{1}{B(a, b)}\log(1-x) \tag{1.5}$$

as $x \to 1$ is due to Gauss, and its refined form

$$B(a, b)F(a, b; a+b; x) + \log(1-x) = R + O((1-x)\log(1-x)) \tag{1.6}$$

as $x \to 1$, with $R = -\Psi(a) - \Psi(b) - 2\gamma$, is due to S. Ramanujan [Ev, p. 553], [Be, p. 71]. Theorems 1.3 and 1.4 are refinements of these classical relations. Ramanujan also gave extensions of (1.6) to the generalized hypergeometric function ${}_pF_q$ for certain values of p and q [Ev, pp. 553–558], [Be, pp. 12, 71]. Formulas (1.5) and (1.6) follow from the identity in [AS, 15.3.10], which has been generalized recently in [B, p. 152].

1.7. **Theorem.** *For each* $a, b \in (0, \infty)$ *the function*

$$f(x) \equiv (1-x)^{1/4}F(a, b; a+b; x)$$

is a strictly decreasing automorphism of $(0, 1)$ *if and only if* $4ab \le a+b$.

While studying relationships between the arithmetic-geometric mean and some other means, J. and P. Borwein [BB2, (2.9)] recently proved that

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x^c\right) < F\left(\frac{1}{2}-\delta, \frac{1}{2}+\delta; 1; 1-x^d\right) \tag{1.8}$$

for all $x \in (0, 1)$, with $c = 2$, $d = 3$, $\delta = \frac{1}{6}$.

We obtain the following generalization.

1.9. **Theorem.** *For* $c, d \in (0, \infty), 4c < \pi d$, *inequality* (1.8) *holds for all* $x \in (0, 1)$ *and for all* $\delta \in (0, \delta_o)$, *where* $\delta_o = ((d\pi - 4c)/(4\pi d))^{1/2}$.

We conclude the paper by proving some inequalities for the gamma function. Our notation is mostly standard. For $x \in [0, 1]$, we often denote $x' = \sqrt{1 - x^2}$.

2. Proofs

For many important properties of the functions $F(a, b; c; x)$, $\mathscr{K}(x)$, and $\mathscr{E}(x)$ see [AS, pp. 556–566, 589–626], [C, Chapters 2, 9], [PBM, pp. 430–615]. Some algorithms for computing these functions are given in [Ba, pp. 345–372, 416–424], [M, Chapter 7], [PNB, p. 249], and [PT, pp. 404–426]. Various aspects of the theory of special functions are surveyed in [Ask1], [Ask2], [AVV3], [LO, 5.3, 5.5].

We shall give here a proof of the Gauss asymptotic formula (1.5), since we have been unable to find a proof in the recent literature or in the standard texts. Various generalizations of (1.5) appear in [Ev, pp. 553–558], [B, p. 152].

2.1. **Lemma.** (1) *For* $a, b \in (0, \infty)$, *the sequence*

$$f(n) \equiv \frac{(a, n)(b, n)}{(a + b, n)(n - 1)!}$$

is increasing to the limit $1/B(a, b)$, *as* $n \to \infty$.

(2) *For* $a, b \in (0, 1)$, *the sequence*

$$g(n) \equiv \frac{(a, n)(b, n)(n + 1)}{(a + b, n)n!}$$

is decreasing to the limit $1/B(a, b)$, *as* $n \to \infty$.

(3) *For* $a, b \in (1, \infty)$, *the sequence* $g(n)$ *is increasing to the limit* $1/B(a, b)$, *as* $n \to \infty$.

In particular, for each positive integer n,

$$\frac{ab}{a + b} \leq f(n) < \frac{1}{B(a, b)} \quad \textit{for all } a, b \in (0, \infty), \tag{4}$$

$$\frac{1}{B(a, b)} < g(n) \leq \frac{2ab}{a + b} \quad \textit{for all } a, b \in (0, 1), \tag{5}$$

$$\frac{2ab}{a + b} \leq g(n) < \frac{1}{B(a, b)} \quad \textit{for all } a, b \in (1, \infty), \tag{6}$$

where the weak inequalities reduce to equality if and only if $n = 1$.

Proof. Part (1) is proved in [AVV3, Theorem 6(4)]. For (2) and (3),

$$\frac{g(n + 1)}{g(n)} = \frac{(a + n)(b + n)(n + 2)}{(a + b + n)(n + 1)^2} < 1$$

iff $w_1 \equiv ((a + b + n)n + ab)(n + 2) < w_2 \equiv (a + b + n)(n + 1)^2$, which is true since $w_2 - w_1 = (a + b - 2ab) + n(1 - ab)$ is positive if $a, b \in (0, 1)$, and negative if $a, b \in (1, \infty)$. The limiting values follow form Stirling's formula [C, p. 24], [Mi, p. 184]. □

2.2. *Remark.* The following inequalities due to Wallis appear in [Mi, p. 192]:

$$\frac{1}{\sqrt{\pi(n+\frac{1}{2})}} < \frac{(\frac{1}{2}, n)}{n!} < \frac{1}{\sqrt{\pi n}}.$$

Since $B(\frac{1}{2}, \frac{1}{2}) = \pi$, the result in Lemma 2.1(4) generalizes the second Wallis inequality here.

2.3. **Lemma** (Gauss). *For* $a, b \in (0, \infty)$, *the relation* (1.5) *holds.*

Proof. Let $B = B(a, b)$. From Lemma 2.1, we have

$$F(a, b; a+b; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(a+b, n)n!} x^n \le 1 + \sum_{n=1}^{\infty} \frac{1}{B} \frac{x^n}{n} = 1 - \frac{1}{B} \log(1-x).$$

Hence

$$\limsup_{x \to 1} \frac{F(a, b; a+b; x)}{\log(1/(1-x))} \le \frac{1}{B}.$$

Next, fix $\varepsilon \in (0, 1/B)$. Then by Lemma 2.1 there exists a positive integer n_o such that $(a, n)(b, n)/((a+b, n)(n-1)!) > (1/B) - \varepsilon$, for all $n \ge n_o$. Hence,

$$F(a, b; a+b; x) \ge F_{n_o} + \left(\frac{1}{B} - \varepsilon\right) \sum_{n=n_o+1}^{\infty} \frac{x^n}{n} = F_{n_o} - \left(\frac{1}{B} - \varepsilon\right) (\log(1-x) + S_{n_o}),$$

where

$$F_{n_o} = \sum_{n=1}^{n_o} \frac{(a, n)(b, n)}{(a+b, n)n!} x^n, \quad S_{n_o} = \sum_{n=1}^{n_o} \frac{x^n}{n}.$$

Dividing by $\log(1/(1-x))$ and letting $x \to 1$, since $\varepsilon \in (0, 1/B)$ is arbitrary, we get

$$\liminf_{x \to 1} \frac{F(a, b; a+b; x)}{\log(1/(1-x))} \ge \frac{1}{B},$$

so that the limit is $1/B$. □

2.4. *Proof of Theorem* 1.3. (1) The limiting value as $x \to 0$ follows from series expansions, while the one as $x \to 1$ follows from Lemma 2.3. Next, let $g(x) = F(a, b; a+b; x) - 1$ and $h(x) = \log(1/(1-x))$. Then $g(0) = h(0) = 0$, and by the monotone l'Hôpital rule [AVV4, Lemma 2.2] it is enough to show that $g'(x)/h'(x)$ is strictly increasing. Now [WW, p. 281]

$$\begin{aligned} \frac{g'(x)}{h'(x)} &= \frac{ab}{a+b}(1-x)F(a+1, b+1; a+b+1; x) \\ &= \frac{ab}{a+b}\left(1 + \sum_{n=0}^{\infty}\left(\frac{(a+1, n+1)(b+1, n+1)}{(a+b+1, n+1)(n+1)!}\right.\right. \\ &\qquad \left.\left. - \frac{(a+1, n)(b+1, n)}{(a+b+1, n)n!}\right) x^{n+1}\right). \end{aligned}$$

The coefficient of x^{n+1} here is positive if and only if $(a+1+n)(b+1+n) > (a+b+n+1)(n+1)$, which holds if and only if $ab + (a+b)(n+1) + (n+1)^2 > (a+b)(n+1) + (n+1)^2$, which is true.

(2) The limiting values are clear from (1.6). Next, by series expansion,

$$g(x) - B = \sum_{n=1}^{\infty} \left(B \frac{(a,n)(b,n)}{(a+b,n)(n-1)!} - 1 \right) \frac{x^n}{n},$$

so that all coefficients are negative by Lemma 2.1(4). □

2.5. *Proof of Theorem* 1.4. For (1), let $g(x) = xF(a,b;c;x)$, $c = a+b$, and $h(x) = \log(1/(1-x))$. Then $g(0) = h(0) = 0$ and

$$\begin{aligned}
\frac{g'(x)}{h'(x)} &= (1-x)\left(F(a,b;c;x) + \frac{abx}{c}F(a+1,b+1;c+1;x)\right) \\
&= (1-x)\left(\sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!} + \frac{ab}{c}\sum_{n=0}^{\infty} \frac{(a+1,n)(b+1,n)}{(c+1,n)} \frac{x^{n+1}}{n!}\right) \\
&= (1-x)\left(\sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(a,n+1)(b,n+1)}{(c,n+1)} \frac{x^{n+1}}{n!}\right) \\
&= \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \left(\frac{(a,n+1)(b,n+1)}{(c,n+1)} - \frac{(a,n)(b,n)}{(c,n)}\right) \frac{x^{n+1}}{n!} \\
&\quad - \sum_{n=0}^{\infty} \frac{(a,n+1)(b,n+1)}{(c,n+1)} \frac{x^{n+2}}{n!}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{g'(x)}{h'(x)} - 1 = &\left(\frac{2ab}{c} - 1\right)x + \sum_{n=0}^{\infty} \frac{(a,n+2)(b,n+2)}{(c,n+2)} \frac{x^{n+2}}{(n+2)!} \\
&+ \sum_{n=0}^{\infty} \frac{(a,n+2)(b,n+2)}{(c,n+2)} \frac{x^{n+2}}{(n+1)!} - \sum_{n=0}^{\infty} \frac{(a,n+1)(b,n+1)}{(c,n+1)} \frac{x^{n+2}}{n!} \\
&- \sum_{n=0}^{\infty} \frac{(a,n+1)(b,n+1)}{(c,n+1)} \frac{x^{n+2}}{(n+1)!} \\
= &\left(\frac{2ab}{c} - 1\right)x + \sum_{n=0}^{\infty} \frac{(a,n+1)(b,n+1)}{(c,n+2)(n+2)!} \\
&\times [(n+1)(ab-1) + (2ab-a-b)]x^{n+2},
\end{aligned}$$

in which all coefficients are negative. Thus $g'(x)/h'(x)$ is decreasing, hence so is $g(x)/h(x)$, by the monotone l'Hôpital rule [AVV4, Lemma 2.2], and (1) follows.

The proof for (2) is similar, except that all coefficients are positive.

Part (3) follows from (1.1) and the series for $\log(1/(1-x))$.

(4) In the series expansion,

$$g_1(x) - (B-1) = \sum_{n=1}^{\infty} \left(\frac{(a,n)(b,n)}{(a+b,n)n!} B - \frac{1}{n+1} \right) x^n,$$

all coefficients are positive by Lemma 2.1(5).

(5) The proof is similar to (4), except that all coefficients are negative by Lemma 2.1(6). The limiting values are clear by (1.6). □

2.6. **Lemma.** *For* $a, b \in (0, \infty)$, $n = 1, 2, 3, \ldots$,

$$\sum_{k=0}^{n} \frac{(a, k)(b, k)}{(a+b, k)k!} > \frac{(a+1, n)(b+1, n)}{(a+b+1, n)n!} > \frac{(a, n)(b, n)}{(a+b, n)n!}.$$

Proof. The first inequality follows by induction, and the second one by the factorial property of (a, n). □

2.7. **Lemma.** *For* $a, b \in (0, \infty)$ *and* $x \in (0, 1)$ *the Maclaurin series of the functions*

$$\frac{1}{1-x} F(a, b; a+b; x) - F(a+1, b+1; a+b+1; x)$$

and

$$F(a+1, b+1; a+b+1; x) - F(a, b; a+b; x)$$

have constant term zero and all other coefficients strictly positive. In particular, these functions are increasing and convex on $(0, 1)$ *and*

$$F(a, b; a+b; x) < F(a+1, b+1; a+b+1; x) < \frac{1}{1-x} F(a, b; a+b; x)$$

for all $x \in (0, 1)$.

Proof. The results follow immediately from Lemma 2.6. □

2.8. *Proof of Theorem* 1.7. The limiting value $f(0) = 1$ is obvious, while $f(1-) = 0$ follows from Theorem 1.3(1). Next, for each $x \in (0, 1)$ we have

$$4(1-x)^{-1/4} f'(x) = \frac{4ab}{a+b} F(a+1, b+1; a+b+1; x) - \frac{1}{1-x} F(a, b; a+b; x).$$

If $4ab \le a+b$, then Lemma 2.7 implies that $f'(x) < 0$. Conversely, if f is decreasing, then $f'(x) \le 0$ for all $x \in (0, 1)$, and letting x tend to 0 we get $4ab \le a+b$. □

The special case $a = b = \frac{1}{2}$ of Theorem 1.7 is well known. This special case follows, for instance, from the infinite product formulas for $x'/\sqrt{x}$ and $(2/\pi)\sqrt{x}\mathscr{K}(x)$ in terms of the Jacobi nome $q = \exp(-\pi\mathscr{K}'/\mathscr{K})$ [WW, p. 488, Exercise 10] or from [AVV1, Theorem 2.2(3)].

2.9. *Remark.* It is easy to show that $(\sqrt{ab}, n)^2 \le (a, n)(b, n) \le ((a+b)/2, n)^2$ for all positive a, b and all $n = 1, 2, \ldots$, with equality if and only if $a = b$. Thus

$$F(\sqrt{ab}, \sqrt{ab}; c; x) \le F(a, b; c; x) \le F((a+b)/2, (a+b)/2; c; x) \tag{2.10}$$

for $a, b, c > 0$, $x \in [0, 1)$, with equality if and only if $a = b$ or $x = 0$. Furthermore, for $a, c > 0$ and for $t \in (0, a)$ we see that $(a+t, n)(a-t, n)$ is a decreasing function of t on $[0, a]$ for $n = 1, 2, \ldots$, so that

$$F(a+t_2, a-t_2; c; x) \le F(a+t_1, a-t_1; c; x) \tag{2.11}$$

for $x \in (0, 1)$, $0 \le t_1 < t_2 \le a$. The second assertion of Lemma 2.7 also follows from the fact that $(a, n)(b, n)/(a+b, n)$, $a, b > 0$, is a strictly increasing function of a.

3. The gamma function

We next study some properties of the functions

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad \text{and} \quad \Psi(x) = \frac{d}{dx} \log \Gamma(x)$$

when x is real and positive.

3.1. **Theorem.** *The function* $f(x) = x(\log x - \Psi(x))$ *is decreasing and convex from* $(0, \infty)$ *onto* $(\frac{1}{2}, 1)$.

Proof. From [WW, p. 251, §12.32, Example] it follows that

$$f(x) = \frac{1}{2} + 2x \int_0^\infty \frac{t\,dt}{(t^2 + x^2)(e^{2\pi t} - 1)}.$$

Hence,

$$\begin{aligned} f'(x) &= 2\int_0^\infty \frac{t\,dt}{(t^2+x^2)(e^{2\pi t}-1)} - 4x^2 \int_0^\infty \frac{t\,dt}{(t^2+x^2)^2(e^{2\pi t}-1)} \\ &= 2\int_0^\infty \frac{t(t^2-x^2)\,dt}{(t^2+x^2)^2(e^{2\pi t}-1)} \\ &= 2\int_0^x \frac{t}{e^{2\pi t}-1} \frac{(t^2-x^2)\,dt}{(t^2+x^2)^2} + 2\int_x^\infty \frac{t}{e^{2\pi t}-1} \frac{(t^2-x^2)\,dt}{(t^2+x^2)^2}. \end{aligned}$$

Since $t/(e^{2\pi t} - 1)$ is decreasing on $(0, \infty)$, we get

$$f'(x) < \frac{2x}{e^{2\pi x} - 1} \int_0^\infty \frac{(t^2 - x^2)\,dt}{(t^2 + x^2)^2}.$$

Substituting $t = x \tan u$, we get

$$\int_0^\infty \frac{(t^2 - x^2)\,dt}{(t^2 + x^2)^2} = \frac{1}{x} \int_0^{\pi/2} (\sin^2 u - \cos^2 u)\,du = 0.$$

Thus we have shown that $f(x)$ is strictly decreasing, so that $f(0+)$ exists. To obtain this limit we observe first that $t/(e^{2\pi t} - 1) < 1/(2\pi)$ implies that

$$\lim_{x \to 0+} 2x \int_0^\infty \frac{t\,dt}{(t^2 + x^2)(e^{2\pi t} - 1)} \le \frac{1}{2}.$$

Next, fix $\varepsilon \in (0, 1/(2\pi))$. Since $\lim_{t \to 0+} t/(e^{2\pi t} - 1) = 1/(2\pi)$, there exists $\delta > 0$ such that $t/(e^{2\pi t} - 1) > 1/(2\pi) - \varepsilon$ for all $t \in (0, \delta)$. Thus

$$2x \int_0^\infty \frac{t\,dt}{(t^2 + x^2)(e^{2\pi t} - 1)} > 2x \left(\frac{1}{2\pi} - \varepsilon\right) \int_0^\delta \frac{dt}{t^2 + x^2} = 2\left(\frac{1}{2\pi} - \varepsilon\right) \arctan \frac{\delta}{x}.$$

Now letting first x, then ε, tend to zero gives

$$\lim_{x \to 0+} 2x \int_0^\infty \frac{t\,dt}{(t^2 + x^2)(e^{2\pi t} - 1)} \ge \frac{1}{2}.$$

For $0 \le a < b \le \infty$ denote

$$I(a, b) = \int_a^b \frac{t\,dt}{(t^2 + x^2)(e^{2\pi t} - 1)}.$$

Then, since $t/(e^{2\pi t}-1) \leq 1/(2\pi)$,

$$2xI(0, 1) \leq \frac{1}{\pi} \arctan \frac{1}{x},$$

while

$$\begin{aligned} I(1, \infty) &\leq \frac{1}{1+x^2} \int_1^{\infty} \frac{t\,dt}{e^{2\pi t}-1} = \frac{1}{1+x^2} \int_1^{\infty} \frac{te^{-\pi t}\,dt}{2\sinh(\pi t)} \\ &\leq \frac{1}{2\pi(1+x^2)} \int_1^{\infty} e^{-\pi t} dt = \frac{1}{2\pi^2(1+x^2)}. \end{aligned}$$

Hence $\lim_{x\to\infty} f(x) = \frac{1}{2}$.

To prove the convexity, let $J[g]$ denote the integral of a function g from 0 to ∞. Then, as above, $f'(x) = 2J[g(t)h(t, x)]$, where

$$g(t) = t/(e^{2\pi t}-1) \quad \text{and} \quad h(t, x) = (t^2-x^2)/(t^2+x^2)^2.$$

Hence,

$$f''(x) = 4xJ[g(t)H(t, x)], \quad \text{where } H(t, x) = \frac{\partial h(t, x)}{\partial x} = \frac{x^2-3t^2}{(t^2+x^2)^3}.$$

Since $g(t)$ is decreasing, by splitting the integral on $(0, \infty)$ into the sum of an integral on $(0, x/\sqrt{3})$ and one on $(x/\sqrt{3}, \infty)$, we get

$$f''(x) > 4xg(x/\sqrt{3})J[H(t, x)].$$

Now substituting $t = x\tan u$, we see that $J[H(t, x)] = 0$. Hence $f''(x) > 0$ on $(0, \infty)$, so that f is convex on $(0, \infty)$. □

3.2. **Theorem.** (1) *The function* $f_1(x) = x^{1/2-x}e^x\Gamma(x)$ *is decreasing and log-convex from* $(0, \infty)$ *onto* $(\sqrt{2\pi}, \infty)$.

(2) *The function* $f_2(x) = x^{1-x}e^x\Gamma(x)$ *is increasing and log-concave from* $(0, \infty)$ *onto* $(1, \infty)$.

Proof. (1) The limiting value at 0 follows from the relation $\Gamma(x+1) = x\Gamma(x)$, while the one at ∞ follows from Stirling's formula. Next,

$$-x\frac{d}{dx}\log f_1(x) = f(x) - \frac{1}{2},$$

where $f(x)$ is as in Theorem 3.1, is clearly positive and decreasing.

(2) The limiting value at 0 follows from the relation $\Gamma(x+1) = x\Gamma(x)$, while the one at ∞ follows from Stirling's formula. Next,

$$-x\frac{d}{dx}\log f_2(x) = 1 - f(x) > 0,$$

where $f(x)$ is as in Theorem 3.1. Moreover, by the monotone l'Hôpital rule [AVV4, §2], it follows that $\frac{d}{dx}\log f_2(x)$ is decreasing. □

3.3. *Remark.* A result similar to Theorem 3.2 appears in [Lu, p. 17], whereas in [Mi, 3.6.55, p. 288] a version of Theorem 3.1 is given. In [G, p. 283] it is shown that $x\Psi(x)$ is convex for $x > 0$. For some recent results on the gamma function see [A1].

4. REFINEMENTS

A problem of interest is to obtain upper estimates for the complete elliptic integral

$$\frac{2}{\pi}\mathscr{K}(x') = F\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x^2\right)$$

in terms of variants of the type

$$F\left(\frac{1}{2}-\delta, \frac{1}{2}+\delta; 1; 1-x^3\right)$$

for $x \in (0, 1)$ and $\delta \in (0, \frac{1}{2})$. Observe initially that the inequality

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x^2\right) < F\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x^3\right) \tag{4.1}$$

for $x \in (0, 1)$ follows immediately from the monotoneity of $F(\frac{1}{2}, \frac{1}{2}; 1; x)$ as a function of x. We seek a refinement of (4.1).

J. Borwein and P. Borwein [BB2, (2.9)] have shown that

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x^2\right) < F\left(\frac{1}{2}-\delta, \frac{1}{2}+\delta; 1; 1-x^3\right) \tag{4.2}$$

for $\delta = \frac{1}{6}$ and for all $x \in (0, 1)$. It has been conjectured recently [AVV3, p. 79] that (4.2) holds for all $\delta \in (0, \frac{1}{6})$ and for all $x \in (0, 1)$. We next obtain a refinement of (4.2), which also proves the statement of Theorem 1.9.

4.3. **Theorem.** *Let $x \in (0, 1)$, $c, d \in (0, \infty)$, $4c < \pi d$. Then*

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x^c\right) < F\left(\frac{1}{2}-\delta_0, \frac{1}{2}+\delta_0; 1; 1-x^d\right) \tag{4.4}$$

$$< F\left(\frac{1}{2}-\delta, \frac{1}{2}+\delta; 1; 1-x^d\right) \tag{4.5}$$

$$< F\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x^d\right) \tag{4.6}$$

for all $\delta \in (0, \delta_0)$, where $\delta_0 = ((d\pi - 4c)/(4\pi d))^{1/2}$.

Proof. Let $f_\delta(x)$ be defined by

$$f_\delta(x) = \frac{1 - F(\frac{1}{2}-\delta, \frac{1}{2}+\delta; 1; x)}{\log(1-x)}. \tag{4.7}$$

It follows from Theorem 1.3 that, for all $x \in (0, 1)$ and $\delta \in [0, \frac{1}{2})$,

$$\left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}+\delta\right) < f_\delta(x) < \frac{1}{B(\frac{1}{2}-\delta, \frac{1}{2}+\delta)}, \tag{4.8}$$

where $B(a, b)$ is the Euler beta function. From the reflection formula $\Gamma(a)\Gamma(1-a) = \pi/\sin(a\pi)$ [AS, p. 256], it follows that

$$\frac{1}{4} - \delta^2 < f_\delta(x) < \frac{1}{\pi}\sin\left(\left(\frac{1}{2}-\delta\right)\pi\right) \tag{4.9}$$

for all $x \in (0, 1)$ and $\delta \in [0, \frac{1}{2})$. From (4.9) with $\delta = 0$ it follows that

$$\frac{F(\frac{1}{2}, \frac{1}{2}; 1; 1 - x^c) - 1}{\log(1/x^c)} = f_0(1 - x^c) < \frac{1}{\pi}.$$

Again applying (4.9) with $\delta \in (0, \delta_0)$ yields

$$\begin{aligned}\frac{F(\frac{1}{2} - \delta, \frac{1}{2} + \delta; 1; 1 - x^d) - 1}{\log(1/x^c)} &= \frac{d}{c}\left[\frac{F(\frac{1}{2} - \delta, \frac{1}{2} + \delta; 1; 1 - x^d) - 1}{\log(1/x^d)}\right] \\ &= \frac{d}{c} f_\delta(1 - x^d) > \frac{d}{c}\left(\frac{1}{4} - \delta^2\right) \\ &> \frac{d}{c}\left(\frac{1}{4} - \delta_0^2\right) = \frac{1}{\pi}.\end{aligned}$$

Therefore, for $0 < \delta \le \delta_0$ and $x \in (0, 1)$,

$$\begin{aligned}F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - x^c\right) &= 1 + f_0(1 - x^c) \log \frac{1}{x^c} \\ &< 1 + \frac{1}{\pi} \log \frac{1}{x^c} \\ &< 1 + \frac{d}{c} f_\delta(1 - x^d) \log \frac{1}{x^c} \\ &= F\left(\frac{1}{2} - \delta, \frac{1}{2} + \delta; 1; 1 - x^d\right),\end{aligned}$$

which establishes (4.4). Inequalities (4.5) and (4.6) are immediate consequences of (2.11). □

4.10. *Conjectures.* (1) For $c = 2$, $d = 3$ the best possible value of δ_0 for which Theorem 4.3 is valid is $\delta_0 = (\pi - 2\arcsin(2/3))/(2\pi) \approx 0.268$ (see [AVV3, p. 6]).

(2) Theorem 1.4 has a counterpart for the generalized hypergeometric function ${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x)$ for the case $a_i > 0$, $b_j > 0$, $p = q + 1$, when the sum is zero-balanced, i.e. when $\sum_{i=1}^p a_i = \sum_{j=1}^q b_j$. See also [B, p. 152].

Acknowledgments

The research of the last-named author was supported by the Academy of Finland. The authors acknowledge useful conversations with K. Pearce.

References

[AS] M. Abramowitz and I. A. Stegun, eds., *Handbook of mathematical functions with formulas, graphs and mathematical tables*, Dover, New York, 1965.

[Al] H. Alzer, *Some gamma function inequalities*, Math. Comp. **60** (1993), 337–346.

[AVV1] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Functional inequalities for complete elliptic integrals and their ratios*, SIAM J. Math. Anal. **21** (1990), 536–549.

[AVV2] ———, *Functional inequalities for hypergeometric functions and complete elliptic integrals*, SIAM J. Math. Anal. **23** (1992), 512–524.

[AVV3] ———, *Hypergeometric functions and elliptic integrals*, Current Topics in Analytic Function Theory (H. M. Srivastava and S. Owa, eds.), World Sci. Publ., Singapore and London, 1992, pp. 48–85.

[AVV4] ——, *Inequalities for quasiconformal mappings in space*, Pacific J. Math. **160** (1993), 1–18.

[Ask1] R. Askey, *Ramanujan and hypergeometric and basic hypergeometric series*, Ramanujan Internat. Symposium on Analysis (December 26–28, 1987) (N. K. Thakare, ed.), Pune, India.

[Ask2] ——, *Handbooks of special functions*, A Century of Mathematics in America, Part III (P. Duren, ed.), Amer. Math. Soc., Providence, RI, 1989, pp. 369–391.

[Ba] L. Baker, *C Mathematical Function Handbook*, McGraw-Hill, New York, 1992.

[Be] B. C. Berndt, *Ramanujan's notebooks*, Part II, Springer-Verlag, Berlin, Heidelberg, and New York, 1989.

[BB1] J. M. Borwein and P. B. Borwein, *Pi and the AGM*, Wiley, New York, 1987.

[BB2] ——, *Inequalities for compound mean iterations with logarithmic asymptotes*, J. Math. Anal. Appl. **177** (1993), 572–582.

[B] W. Bühring, *Generalized hypergeometric functions at unit argument*, Proc. Amer. Math. Soc. **114** (1992), 145–153.

[BF] P. F. Byrd and M. D. Friedman, *Handbook of elliptic integrals for engineers and physicists*, Grundlehren Math. Wiss., vol. 57, Springer-Verlag, Berlin, Göttingen and Heidelberg, 1954.

[C] B. C. Carlson, *Special functions of applied mathematics*, Academic Press, New York, 1977.

[Ev] R. J. Evans, *Ramanujan's second notebook: Asymptotic expansions for hypergeometric series and related functions*, Ramanujan Revisited: Proc. of the Centenary Conference (Univ. of Illinois at Urbana-Champaign) (G. E. Andrews, R. A. Askey, B. C. Berndt, R. G. Ramanathan, and R. A. Rankin, eds.), Academic Press, Boston, 1988, pp. 537–560.

[G] W. Gautschi, *Some mean value inequalities for the gamma function*, SIAM J. Math. Anal. **5** (1974), 282–292.

[LO] D. W. Lozier and F. W. J. Olver, *Numerical evaluation of special functions*, Proc. Sympos. Appl. Math. (to appear).

[Lu] Yu. L. Luke, *Mathematical functions and their approximations*, Academic Press, New York, 1975.

[Mi] D. S. Mitrinović, *Analytic inequalities*, Grundlehren Math. Wiss., Band 165, Springer-Verlag, Berlin, Göttingen, and Heidelberg, 1970.

[M] S. L. Moshier, *Methods and programs for mathematical functions*, Ellis Horwood, Chichester, 1989.

[PNB] W. F. Perger, M. Nardin, and A. Bhalla, *A numerical evaluator for the generalized hypergeometric function*, Comput. Phys. Comm. **77** (1993), 249–254.

[PT] B. A. Popov and G. S. Tesler, *Computation of functions on electronic computers—Handbook*, Naukova Dumka, Kiev, 1984. (Russian)

[PBM] A. P. Prudnikov, Yu. A. Brychkov, and O. Marichev, *Integrals and series*, Vol. 3: *More special functions*, transl. from the Russian by G. G. Gould, Gordon and Breach, New York, 1988.

[WW] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, 4th ed., Cambridge Univ. Press, London and New York, 1958.

Department of Mathematics, Michigan State University, East Lansing, Michigan 48824

E-mail address: `anderson@math.msu.edu`

Department of Mathematics, Texas Tech University, Lubbock, Texas 79409

E-mail address: `narwb@ttacs1.ttu.edu`

Department of Mathematics, Southwestern University, Georgetown, Texas 78626

E-mail address: `richardk@ralph.txswk.edu`

Department of Mathematics, University of Auckland, Auckland, New Zealand

E-mail address: `vamanamu@mat.auckland.ac.nz`

Department of Mathematics, University of Helsinki, FIN-00100 Helsinki, Finland

E-mail address: `vuorinen@csc.fi`